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Towards the ultimate variance-conserving convection scheme

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Abstract

In the past various arguments have been used for applying kinetic energy-conserving advection schemes in numerical simulations of incompressible fluid flows. One argument is obeying the programmed dissipation by viscous stresses or by sub-grid stresses in Direct Numerical Simulation and Large Eddy Simulation, see e.g. [Phys. Fluids A 3 (7) (1991) 1766]. Another argument is that, according to e.g. [J. Comput. Phys. 6 (1970) 392; 1 (1966) 119], energy-conserving convection schemes are more stable i.e. by prohibiting a spurious blow-up of volume-integrated energy in a closed volume without external energy sources. In the above-mentioned references it is stated that nonlinear instability is due to spatial truncation rather than to time truncation and therefore these papers are mainly concerned with the spatial integration. In this paper we demonstrate that discretized temporal integration of a spatially variance-conserving convection scheme can induce non-energy conserving solutions. In this paper the conservation of the variance of a scalar property is taken as a simple model for the conservation of kinetic energy. In addition, the derivation and testing of a variance-conserving scheme allows for a clear definition of kinetic energy-conserving advection schemes for solving the Navier–Stokes equations. Consequently, we first derive and test a strictly variance-conserving space–time discretization for the convection term in the convection–diffusion equation. Our starting point is the variance-conserving spatial discretization of the convection operator presented by Piacsek and Williams [J. Comput. Phys. 6 (1970) 392]. In terms of its conservation properties, our variance-conserving scheme is compared to other spatially variance-conserving schemes as well as with the non-variance-conserving schemes applied in our shallow-water solver, see e.g. [Direct and Large-eddy Simulation Workshop IV, ERCOFTAC Series, Kluwer Academic Publishers, 2001, pp. 409–287].

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1. Introduction

Natural water flows appear in many civil-engineering applications and these free-surface flows are nearly two-dimensional. For these flows, we aim at simulating the largest and energetic horizontal

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turbulence structures through a procedure that we call horizontal large eddy simulation (HLES), see [7].

Compared to the forcing of the mean flow, the interaction between turbulence structures at different length scales is a slow and dynamically weak process in which the conservation of energy, while being redistributed among wave numbers, is essential. It is our prime motivation to focus on energy-conserving schemes while we acknowledge that, e.g. momentum conservation or treatment of shocks or bores may be more important in other applications.

Presently, for the simulation of the free-surface flows mentioned above, we apply our general-purpose shallow-water solver that is based on ADI temporal integration combining explicit second-order central and implicit second- or third-order upwind schemes for the advection or convection operators. These schemes have good robustness properties but lower odd-order upwind discretizations are known to be dissipative, as experienced in [2] and particularly exhibited in frequency spectra in [7]. The vast experience of others in DNS and LES recommends the application of discrete convection operators that conserve kinetic energy. The latter is the objective of this paper.

Below we motivate why we start with deriving variance-conserving convection schemes and end with kinetic-energy conserving advection scheme.

Variance is defined as the quadratic quantity of a scalar, $\frac{1}{2}\phi^2$. It can be considered as a model for the kinetic energy $\frac{1}{2}\vec{u}^2(=\frac{1}{2}u^2+\frac{1}{2}v^2+\frac{1}{2}w^2)$, since conservation of variance is equivalent to conservation of the square of each of the Cartesian velocity components, $\frac{1}{2}u^2$, $\frac{1}{2}v^2$ and $\frac{1}{2}w^2$, which guarantees the conservation of kinetic energy. For simplicity, clarity in notation as well as for a clear definition of essential time levels, we consider the convection of a scalar yielding a clear notational distinction between the flux terms and the convected quantity.

This paper derives a numerical scheme for the convection term in the convection–diffusion equation, that is variance-conserving in space as well as in time. In earlier publications – e.g. [1,4,5] – a spatially, but not temporally variance-conserving scheme for the convection terms is derived and that scheme is called skew-symmetric [1,3]. When this scheme is used in inviscid flow simulations, it still violates the conservation of kinetic energy, and in [5] this violation is attributed to the time-stepping method. Another complication is that Morinishi et al. [5] examined the convection of velocity, without making a notational distinction between the flux terms (velocity) and the convected quantity (velocity). This causes ambiguities in the interpretation of the time integration and the time levels of the various quantities in the discrete equations. By bringing into light the difference between the flux terms and the convected quantity, we designed another procedure for obtaining a numerical scheme for the convection terms that conserves variance in space as well as in time. For clarity reasons, we confine ourselves to the two-dimensional case. The three-dimensional extension of our scheme is a simple extension of the two-dimensional one.

2. Definition of various continuous forms of the convection operator

Analog to the definitions in [5], this section repeats three different continuous representations for the convection operator.

The linear convection–diffusion equation can be written symbolically as

$$\frac{\partial \phi}{\partial t} + (Conv) = (Diff) \tag{2.1}$$

with

$$(Diff) \equiv \kappa \nabla^2 \phi, \tag{2.2}$$

where ϕ is the scalar and κ is the diffusion coefficient. In Eq. (2.1), $(Conv)$ implies a generic form of the convection operator as defined subsequently.

The linear term (*Conv*) is part of the material derivative of ϕ and expressed in two-dimensional Cartesian co-ordinates (x, y) with (u, v) velocity components it reads

$$(Conv) \Rightarrow (Adv) \equiv u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} = \vec{u} \cdot \vec{\nabla}\phi. \tag{2.3}$$

For an incompressible medium we can also write

$$(Conv) \Rightarrow (Div) \equiv \frac{\partial u\phi}{\partial x} + \frac{\partial v\phi}{\partial y} = \vec{\nabla} \cdot (\vec{u}\phi), \tag{2.4}$$

or

$$(Conv) \Rightarrow (Skew) \equiv \frac{1}{2}\left(u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y}\right) + \frac{1}{2}\left(\frac{\partial u\phi}{\partial x} + \frac{\partial v\phi}{\partial y}\right) = \frac{1}{2}(Adv) + \frac{1}{2}(Div), \tag{2.5}$$

since for an incompressible flow the continuity equation $\vec{\nabla} \cdot \vec{u} = 0$ holds. Here, (*Adv*) is referred to as the *advection* form, (*Div*) is the *divergence* form and (*Skew*) the *skew-symmetric* form. These definitions agree with those in [5].

Gauss' theorem states that a transport operator is spatially conservative if it can be expressed in the divergence form. In the following, we will show that the skew-symmetric form conserves variance of the scalar ϕ, irrespective of whether $\vec{\nabla} \cdot \vec{u} = 0$ holds.

The transport equation for the variance results from multiplying Eq. (2.1) with ϕ:

$$\phi\frac{\partial \phi}{\partial t} + \phi(Conv) = \phi(Diff) \Rightarrow \frac{\partial \frac{1}{2}\phi^2}{\partial t} + \phi(Conv) = \phi(Diff). \tag{2.6}$$

Next, for the three different forms, we focus on the convection operator multiplied with ϕ:

$$\phi(Adv) \equiv \phi\left(u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y}\right) = \vec{\nabla} \cdot \left(\vec{u}\frac{1}{2}\phi^2\right) - \frac{1}{2}\phi^2(\vec{\nabla} \cdot \vec{u}), \tag{2.7}$$

$$\phi(Div) \equiv \phi\left(\frac{\partial u\phi}{\partial x} + \frac{\partial v\phi}{\partial y}\right) = \vec{\nabla} \cdot \left(\vec{u}\frac{1}{2}\phi^2\right) + \frac{1}{2}\phi^2(\vec{\nabla} \cdot \vec{u}), \tag{2.8}$$

$$\phi(Skew) \equiv \phi\left\{\frac{1}{2}(Adv) + \frac{1}{2}(Div)\right\} = \vec{\nabla} \cdot \left(\vec{u}\frac{1}{2}\phi^2\right) \tag{2.9}$$

We can now conclude the following. Eq. (2.9) shows that the skew-symmetric form of the convection operator can be written in divergence form in the transport equation for the variance of the scalar ϕ. Therefore, the skew-symmetric form conserves variance, irrespective of the presence of divergence of the velocity vector, $\vec{\nabla} \cdot \vec{u}$. In the continuous case, $\vec{\nabla} \cdot \vec{u} = 0$ holds for an incompressible flow. However, for compressible flows, for the depth-averaged shallow-water equations (free-surface flows) or for ill-converged pressure corrections, $\vec{\nabla} \cdot \vec{u} \neq 0$ holds. For these cases the variance-conserving properties of the advection and divergence form of the convection operator are not guaranteed. Even if in the continuous case $\vec{\nabla} \cdot \vec{u} \neq 0$ holds, then in finite difference forms the divergence may not necessarily be zero at all times. If the latter deviation occurs the discretized versions of the advective and divergence form of the convection term violate the conservation of variance where the skew-symmetric does not.

In the next section, the existing spatially variance-conserving convection scheme [1] is written in our notation, aiming at a clear distinction between the flux terms $\vec{u}$ and the transported quantity ϕ.

3. Existing skew-symmetric spatial discretization

Piacsek and Williams [1] propose a skew-symmetric discretization for the convection term on a staggered grid. This scheme is spatially variance-conserving and has the following form for grid point (i, j), see also Fig. 1:

$$(Skew)_{\Delta,(i,j)} \equiv \frac{u_{i+1/2,j}\phi_{i+1,j} - u_{i-1/2,j}\phi_{i-1,j}}{2\Delta x} + \frac{v_{i,j+1/2}\phi_{i,j+1} - v_{i,j-1/2}\phi_{i,j-1}}{2\Delta y}, \tag{3.1}$$

where the subscript $\Delta, (i, j)$ denotes the discrete approximation of $(Skew)$ in the grid-point (i, j).

Fig. 1 shows a two-dimensional grid and a computational cell with the staggered arrangement of the velocity vectors and scalar positions. For clarity reasons we restrict ourselves to the two-dimensional case. The three-dimensional scheme is a simple extension of the two-dimensional scheme. In addition, we consider just equidistant grids.

In [5] and its notation, it is stated that a term is conservative in the discrete system if it can be written as

$$Q(\phi) = \frac{\delta_1\left({}^1F_j(\phi)\right)}{\delta_1 x_j} + \frac{\delta_2\left({}^2F_j(\phi)\right)}{\delta_2 x_j} + \cdots \tag{3.2}$$

where, in [5], the finite-difference operator (second-order central) with respect to x is defined as

$$\left.\frac{\delta_n \phi}{\delta_n x}\right|_{x,y} = \frac{\phi(x + n\Delta x/2, y) - \phi(x - n\Delta x/2, y)}{n\Delta x}. \tag{3.3}$$

Morinishi et al. [5] show that the discrete skew-symmetric form of the convection operator, analog to the definition in Eq. (3.1), can be rewritten in the form (3.2) in the transport equation of the kinetic energy. However, Morinishi et al. experience by numerical experiments that using this form for the convection terms in the Navier–Stokes equations still violates the conservation of kinetic energy. They conclude that this violation is due to their time integration method (third-order Runge–Kutta).

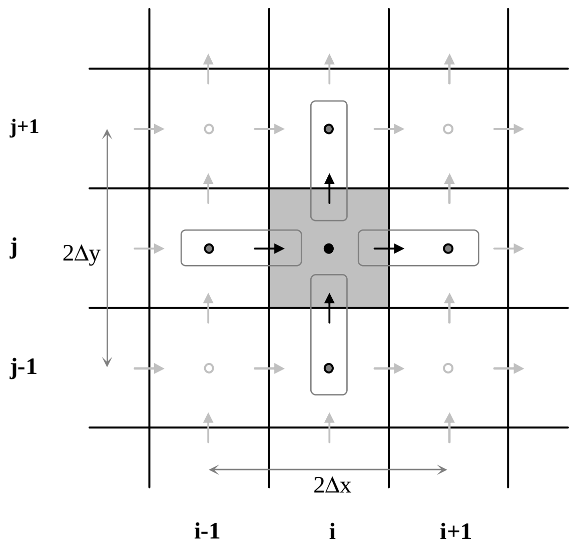

Fig. 1. 2D computational cell of the skew-symmetric convection of a scalar property.

In the following section, we present a numerical scheme for the convection operator that is variance-conserving not only in space but in time as well. This is a new scheme in which we strive for a clear definition of the time levels of the flux terms and the convected quantity.

4. Derivation of space–time variance-conserving convection scheme

An essential difference between previously cited papers and our derivation below is the following. We derive the space–time variance-conserving transport scheme directly from the discretized transport equation for the variance of a scalar quantity in absence of diffusion, sources and sinks. This is in contrast to the method of for example Piacsek and Williams [1], who use the existing advective and divergence dicretization form of the convection operator in the transport equation for a scalar quantity to derive their spatial variance-conserving scheme.

For deriving our transport scheme, conservative both in space as well as in time – in the following referred to as 'the space–time variance-conserving scheme' – we must include time levels in the scalar multiplication of Eq. (2.6). The latter multiplication is repeated below. In the following, we focus solely on the convection term and omit the diffusion term:

$$\phi\left\{\frac{\partial\phi}{\partial t}+(Conv)=0\right\}\Rightarrow\frac{\partial\frac{1}{2}\phi^2}{\partial t}+\phi(Conv)=0. \tag{2.6}$$

This multiplication is repeated below on a discrete space–time level, restricted to a single spatial dimension while using the second-order central time discretization in a single-step fashion:

$$\left(\frac{\partial\phi}{\partial t}\right)_i^{n+1/2}\approx\frac{\phi_i^{n+1}-\phi_i^n}{\Delta t}=-(Conv)_{\Delta,i}. \tag{4.1}$$

For obtaining the discretized transport equation (2.6) for the variance $\frac{1}{2}\phi^2$, Eq. (4.1) is multiplied with some discrete multiplier $\phi_{*,i}$, where the first subscript refers to a yet unknown time level but this level follows from the following construction. First, we demand that the following multiplication:

$$\phi_{*,i}\left\{\frac{\phi_i^{n+1}-\phi_i^n}{\Delta t}= -(Conv)_{\Delta,i}\right\}$$

should comply with

$$\frac{\frac{1}{2}(\phi_i^{n+1})^2-\frac{1}{2}(\phi_i^n)^2}{\Delta t}=-\phi_{*,i}(Conv)_{\Delta}. \tag{4.2}$$

From comparing the left-hand side of (4.2) with the former equation follows that the discrete multiplier $\phi_{*,i}$ is to be defined as

$$\phi_{*,i}\equiv\frac{\phi_i^{n+1}+\phi_i^n}{2}. \tag{4.3}$$

Eq. (4.3) shows clearly that both the position in space as well as in time of the values of ϕ used in the definition of $\phi_{*,i}$ are essential and this restricts the definition of the convection operator because

$$\phi_{*,i}(Conv)_{\Delta,i}=\frac{\phi_i^{n+1}+\phi_i^n}{2}(Conv)_{\Delta,i} \tag{4.4}$$

must be written in divergence form for obtaining a variance-conserving flux, i.e.

$$\phi_{*,i}(Conv)_{\Delta,i} = \left\{\vec{\nabla}\cdot\left(\vec{u}\frac{1}{2}\phi^2\right)\right\}_{\Delta,i} \tag{4.5}$$

must hold. Consequently, for $(Conv)_{\Delta,i}$ we define

$$(Conv)_{\Delta,i} = \frac{1}{\phi_{*,i}}\left\{\vec{\nabla}\cdot\left(\vec{u}\frac{1}{2}\phi^2\right)\right\}_{\Delta,i}, \tag{4.6}$$

where the undesired division by $\phi_{*,i}$ should be eliminated through the following definition of the convection operator.

Using the second-order central discretization form for $(Conv)_{\Delta,i}$ in Eq. (4.5), we obtain

$$\phi_{*,i}(Conv)_{\Delta,i} = \frac{u_{i+1/2}\frac{1}{2}\phi^2_{i+1/2} - u_{i-1/2}\frac{1}{2}\phi^2_{i-1/2}}{\Delta x} = \frac{u_{i+1/2}}{2\Delta x}\phi^2_{i+1/2} - \frac{u_{i-1/2}}{2\Delta x}\phi^2_{i-1/2}. \tag{4.7}$$

In view of the undesired division by $\phi_{*,i}$ we want to extract $\phi_{*,i}$ from both terms on the right-hand side of Eq. (4.7), i.e.

$$\begin{aligned}\phi^2_{i+1/2} &\equiv \phi_{*,i}\cdot\Phi^{i(+)} = \frac{\phi_i^{n+1}+\phi_i^n}{2}\cdot\Phi^{i(+)},\\ \phi^2_{i-1/2} &\equiv \phi_{*,i}\cdot\Phi^{i(-)} = \frac{\phi_i^{n+1}+\phi_i^n}{2}\cdot\Phi^{i(-)}.\end{aligned} \tag{4.8}$$

The form of the two unknown variables $\Phi^{i(+)}$ and $\Phi^{i(-)}$ is dictated by the second requirement for conservation, namely the cancellation of the flux defined at the faces of adjacent cells. Additionally to the definition in (4.8), the following must hold when shifting the operators one Δx to the negative x-direction:

$$\begin{aligned}\phi^2_{i-1/2} &\equiv \phi_{*,i-1}\cdot\Phi^{i-1(+)} = \frac{\phi_{i-1}^{n+1}+\phi_{i-1}^n}{2}\cdot\Phi^{i-1(+)},\\ \phi^2_{i-3/2} &\equiv \phi_{*,i-1}\cdot\Phi^{i-1(-)} = \frac{\phi_{i-1}^{n+1}+\phi_{i-1}^n}{2}\cdot\Phi^{i-1(-)}.\end{aligned} \tag{4.9}$$

The requirements expressed by (4.8) and (4.9) yield

$$\Phi^{i(-)} = \phi_{*,i-1} = \frac{\phi_{i-1}^{n+1}+\phi_{i-1}^n}{2}. \tag{4.10}$$

and likewise

$$\Phi^{i(+)} = \phi_{*,j+1} = \frac{\phi_{i+1}^{n+1}+\phi_{i+1}^n}{2}. \tag{4.11}$$

Consequently, the substitution of (4.8) into (4.7) and subsequently dividing by $\phi_{*,i}$ yields our definition of the variance-conserving space–time discretization of the convection operator ($Conv$):

$$(Conv)_{\Delta,i} = \frac{u_{i+1/2}\frac{(\phi_{i+1}^{n+1}+\phi_{i+1}^n)}{2} - u_{i-1/2}\frac{(\phi_{i-1}^{n+1}+\phi_{i-1}^n)}{2}}{2\Delta x} = \frac{u_{i+1/2}(\phi_{i+1}^{n+1}+\phi_{i+1}^n) - u_{i-1/2}(\phi_{i-1}^{n+1}+\phi_{i-1}^n)}{4\Delta x}. \tag{4.12}$$

Eq. (4.12) presents our new discrete space–time variance-conserving convection operator. Apparently, the time level of the flux terms u is of no relevance to the conservation of variance and can thus be chosen arbitrarily, if the purpose is conservation of variance. This conclusion is essential because extension of Eq. (4.12) to $\vec{u}\cdot\vec{\nabla}\vec{u}$ allows for solving a linear set of discretized momentum equations. In addition, our scheme (4.12) is derived directly from the discretized transport equation for scalar variance. The latter invokes essential definitions for the time levels in (4.12) and it is this neglect of time levels in previously published spatially variance-conserving schemes that warranted us whether these schemes are variance-conserving in time.

Note that the time-centred average of the convected quantity in the space–time variance-conserving operator (4.12) is the single but essential difference with respect to the skew-symmetric form (3.1) of Piacsek and Williams [1]. In other words, the variance-conserving space–time discretization, as defined in Eq. (4.12), represents a Crank–Nicholson time integration and thus involves the implicit solution of

$$\frac{\phi_i^{n+1}-\phi_i^n}{\Delta t} = -\frac{u_{i+1/2}\left(\phi_{i+1}^{n+1}+\phi_{i+1}^n\right)-u_{i-1/2}\left(\phi_{i-1}^{n+1}+\phi_{i-1}^n\right)}{4\Delta x} \Rightarrow \frac{\phi_i^{n+1}-\phi_i^n}{\Delta t} = -\left[\frac{1}{2}\frac{u_{i+1/2}\phi_{i+1}^{n+1}-u_{i-1/2}\phi_{i-1}^{n+1}}{2\Delta x}+\frac{1}{2}\frac{u_{i+1/2}\phi_{i+1}^{n}-u_{i-1/2}\phi_{i-1}^{n}}{2\Delta x}\right], \tag{4.13}$$

or extended to two dimensions

$$\frac{\phi_{i,j}^{n+1}-\phi_{i,j}^n}{\Delta t} = -\frac{u_{i+1/2,j}\left(\phi_{i+1,j}^{n+1}+\phi_{i+1,j}^n\right)-u_{i-1/2,j}\left(\phi_{i-1,j}^{n+1}+\phi_{i-1,j}^n\right)}{4\Delta x} - \frac{v_{i,j+1/2}\left(\phi_{i,j+1}^{n+1}+\phi_{i,j+1}^n\right)-v_{i,j-1/2}\left(\phi_{i,j-1}^{n+1}+\phi_{i,j-1}^n\right)}{4\Delta y}. \tag{4.14}$$

The previous procedure, especially Eqs. (4.8)–(4.11), shows that the time levels used in the convection operator must equal those used in ϕ_* as well as those applied in the discretized temporal derivative. This implies that a strictly variance-conserving numerical scheme, at least derived by our procedure using the transport Eq. (2.6) for $\frac{1}{2}\phi^2$, will always be implicit. In other words, using the procedure described in this section, it is not possible to define a discretization in space alone that is strictly variance conserving in time. One always has to include the time levels in the discretization of the convective terms and therefore the time integration method is fixed simultaneously.

In the following section we discuss with more detail the complication with respect to the time levels needed for obtaining the space–time variance-conserving convection operator. In view of its feasibility for our shallow-water solver, as given in the introduction to this paper, we investigate ADI for (4.14) and explicit time integration as fair alternatives/compromises.

5. Approximations to the space–time variance-conserving convection scheme

The first part of this section clarifies why explicit temporal integration cannot yield variance conservation in time, at least from the viewpoint of the derivation strategy presented in the previous section. The second part of this section discusses various approximations to (4.14) with the objective of seeking a compromise between accuracy in variance conservation, stability and computational costs.

An explicit method that is strictly variance-conserving has not been found and this can be explained by the following figures.

Fig. 2(a) presents the values of ϕ used in the definition of the variance-conserving convection scheme. It shows that the values needed to define $\phi_{i-1/2}^2$ are identical for Eq. (4.8), which gives the definition on

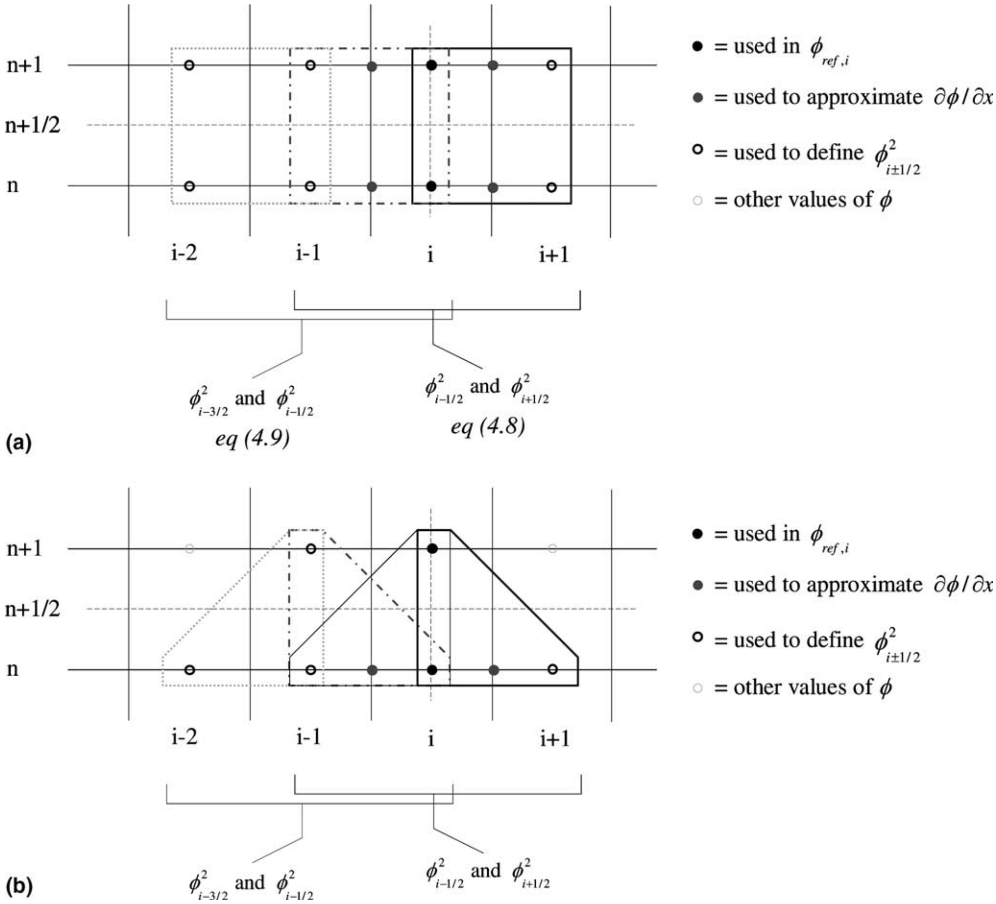

Fig. 2. (a) Values of ϕ used in the space–time variance-conserving scheme. (b) Values of ϕ used in Euler forward explicit scheme.

$x = i$, and Eq. (4.9), which gives the definition on $x = i - 1$, namely the values of ϕ that are enclosed in the square . When the same is done for the Euler forward explicit scheme, in Fig. 2(b), we see that the values needed to define $\phi^2_{i-1/2}$ are not identical. They are enclosed by for the definition on $x = i$ and enclosed by for the definition on $x = i - 1$. Therefore, in the Euler forward explicit scheme, the definition of $\phi^2_{i-1/2}$ differs for fluxes of adjacent cells, which does not yield a variance-conserving definition in time. The previous conclusion holds for any other explicit temporal integration scheme and this ends the first part of this section.

In the remaining part of this section we present our considerations for finding approximations to the space–time variance-conserving form of the transport equation.

The formal approach is solving (4.14), i.e. Crank–Nicholson integration, directly. The set (4.14) is unconditionally stable, the single limitation on the time step is dictated by the wish to avoid errors in variance conservation by amplification of machine inaccuracy.

The computationally less costly alternative is using an iterative solver in the Crank–Nicholson integration (4.14) and because (4.14) is unconditionally stable, the truncation errors due to the convergence criterion on this solver may only endanger variance conservation in time. Nevertheless, the next section shows that the latter danger is not dominant and it appears that, for fair convergence conditions, just machine accuracy matters.

In two dimensions, ADI is unconditionally stable and it yields just tridiagonal systems that are solved directly and nearly as efficient as explicit methods. The difference in variance conservation by ADI and by solving (4.14) is the price to pay for this compromise. In view of the following alternatives the compromise of using ADI rather than (4.14) appears optimal for implementation in our shallow-water solver. In this solver we already use ADI for solving the coupled set of mass-conservation and momentum equations, see also the introduction to this paper.

In contrast to [5] we do not even consider Runge–Kutta integration because of its ambiguity of the boundary conditions on the non-physical velocity field at the intermediate fractional time levels. The reason is that in shallow-water applications, we allow a multitude of open boundary conditions including weakly reflective or Riemann conditions for long-wave motions (tides).

In DNS and LES, the second- and third-order explicit Adams–Bashforth schemes are popular. However, these schemes are in principle dissipative as well as unstable for inviscid flows or impose time-step limitations depending on the subgrid-scale eddy viscosity in (H)LES. We do not favor a turbulence-dependent and thus a time-dependent time step in view of our available post-processing data analysis as well as the coupling strategy to e.g. our off-line biochemical codes. Nevertheless, we apply the second-order explicit Adams–Bashforth scheme in a non-diffusive simulation just to demonstrate that variance conservation does not guarantee stability, as suggested in [1].

An equally unattractive alternative would be the explicit leap-frog scheme although it is non-dissipative but stable only for inviscid flows and may require temporal smoothing of wiggles that destroy mass conservation in our simulations. The leap-frog scheme in a non-diffusive simulation is included here for comparison only.

The leap-frog, the two-step Adams–Bashforth and the ADI time-integration schemes are defined below and together with the Crank–Nicholson scheme in (4.14), these are used in the demonstration presented in the next section.

The Crank–Nicholson scheme in Eq. (4.14) can be written symbolically as

$$\frac{\phi_{i,j}^{n+1}-\phi_{i,j}^{n}}{\Delta t}\approx\left(\frac{\partial\phi}{\partial t}\right)_{i,j}^{n+1/2}=\frac{1}{2}C_{i,j}^{n+1}+\frac{1}{2}C_{i,j}^{n} \tag{5.1}$$

with

$$C_{i,j}^{k}=-\frac{u_{i+1/2,j}\phi_{i+1,j}^{k}-u_{i-1/2,j}\phi_{i-1,j}^{k}}{2\Delta x}-\frac{v_{i,j+1/2}\phi_{i,j+1}^{k}-v_{i,j-1/2}\phi_{i,j-1}^{k}}{2\Delta y},\quad k=0,1,2\ldots,n_{\max}.$$

The leap-frog scheme then reads

$$\frac{\phi_{i,j}^{n+1}-\phi_{i,j}^{n-1}}{2\Delta t}\approx\left(\frac{\partial\phi}{\partial t}\right)_{i,j}^{n}=C_{i,j}^{n}, \tag{5.2}$$

being equivalent to the leap-frog scheme used in [1].

The two-step Adams–Bashforth scheme is defined as follows:

$$\frac{\phi_{i,j}^{n+1}-\phi_{i,j}^{n}}{\Delta t}\approx\left(\frac{\partial\phi}{\partial t}\right)_{i,j}^{n+1/2}=\frac{3}{2}C_{i,j}^{n}-\frac{1}{2}C_{i,j}^{n-1}, \tag{5.3}$$

which is equivalent to the one-leg method derived in [4] for a linear problem. Both for the leap-frog scheme and the two-step Adams–Bashforth scheme the initiation at $n=0$ was carried out with an implicit Euler forward scheme, by setting $\phi_{i,j}^{-1}=\phi_{i,j}^{0}$ and $C_{i,j}^{-1}=C_{i,j}^{0}$, respectively.

The ADI scheme is constructed as follows. At the first half time-step, the x-direction of the convection term is taken at the implicit time-level and the y-direction at the explicit level:

$$\frac{\phi_{i,j}^{n+1/2} - \phi_{i,j}^{n}}{\frac{1}{2}\Delta t} = -\frac{u_{i+1/2,j}\phi_{i+1,j}^{n+1/2} - u_{i-1/2,j}\phi_{i-1,j}^{n+1/2}}{2\Delta x} - \frac{v_{i,j+1/2}\phi_{i,j+1}^{n} - v_{i,j-1/2}\phi_{i,j-1}^{n}}{2\Delta y}. \tag{5.4}$$

At the second half time-step, the time levels are interchanged:

$$\frac{\phi_{i,j}^{n+1} - \phi_{i,j}^{n+1/2}}{\frac{1}{2}\Delta t} = -\frac{u_{i+1/2,j}\phi_{i+1,j}^{n+1/2} - u_{i-1/2,j}\phi_{i-1,j}^{n+1/2}}{2\Delta x} - \frac{v_{i,j+1/2}\phi_{i,j+1}^{n+1} - v_{i,j-1/2}\phi_{i,j-1}^{n+1}}{2\Delta y}. \tag{5.5}$$

The following section presents and discusses a numerical test with these schemes.

6. Numerical test: forced convection of a scalar

For the purpose of demonstration, we consider the forced convection of a scalar ϕ by a two-dimensional frozen velocity field in a closed square domain, which is described by a linear, variable coefficient, partial differential equation for ϕ:

$$\frac{\partial\phi}{\partial t} + U(x,y)\frac{\partial\phi}{\partial x} + V(x,y)\frac{\partial\phi}{\partial y} = 0, \tag{6.1}$$

where

$$\begin{aligned} U &= U_0 \sin(N\pi x)\cos(N\pi y), \\ V &= -U_0 \cos(N\pi x)\sin(N\pi y). \end{aligned} \tag{6.2}$$

Here U_0 has a constant value and the parameter N can be used to vary the number of vortices that are represented by Eq. (6.2). The use of capitals for the velocity components indicates the time-independence of the velocity field.

Eq. (6.1) describes the process of the continuous roll up of a scalar distribution and transport of its spectral energy density to the large wave numbers in the wave-spectrum, up to the limit of the spatial resolution. By using the space–time variance-conserving scheme in this specific equation, we put it to the ultimate test, because convection is the only mechanism present and the variance-spectrum can be evaluated from the smallest to the largest resolvable wave numbers.

If the velocity field $\vec{u}$ and the scalar ϕ each have different wave numbers or are based on more than a single wave number then temporal integration of the convective operator $\vec{u}\cdot\vec{\nabla}\phi$ or the advective operator $\vec{u}\cdot\vec{\nabla}\vec{u}$ redistributes energy in wave number space while conserving the volume-integrated energy, see e.g. [8]. Through temporal integration, a part of the original spectral-energy density contained in $\vec{u}$ or ϕ is transferred to increasingly larger wave numbers. The latter process then produces increasingly sharper gradients in space. This process holds for both the convective or the advective operator and for simplicity therefore this paper uses the $\vec{u}\cdot\vec{\nabla}\phi$ as illustration.

The velocity components in Eq. (6.2) satisfy $\vec{\nabla}\cdot\vec{U} = 0$ in their continuous form, but upon finite differencing introduce divergence errors $\sim 10^{-7}$ on the unit square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, covered by an equidistant 51×51 mesh, for $U_0 = 2$.

In view of Eq. (4.14), the space–time variance-conserving scheme for Eq. (6.1) reads

$$\frac{\phi_{i,j}^{n+1}-\phi_{i,j}^{n}}{\Delta t}+\frac{U_{i+1/2,j}\left(\phi_{i+1,j}^{n+1}+\phi_{i+1,j}^{n}\right)-U_{i-1/2,j}\left(\phi_{i-1,j}^{n+1}+\phi_{i-1,j}^{n}\right)}{4\Delta x}+\frac{V_{i,j+1/2}\left(\phi_{i,j+1}^{n+1}+\phi_{i,j+1}^{n}\right)-V_{i,j-1/2}\left(\phi_{i,j-1}^{n+1}+\phi_{i,j-1}^{n}\right)}{4\Delta y}=0. \tag{6.3}$$

Eq. (6.3) is solved on the unit square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, with closed boundaries and covered by an equidistant 51×51 staggered grid. We set the flux through the boundaries equal to zero. In the simulations, $N = 1$ in Eq. (6.2) is used, yielding four counter-rotating vortices on the square domain.

The volume-integrated variance is computed at every time step and compared to its initial value. These are defined by

$$\mathrm{var}(t)=\frac{1}{M}\sum_{i,j}\frac{1}{2}\phi_{i,j}^{2}(t), \quad \mathrm{var}(t=0)=\mathrm{var}(0)=\frac{1}{M}\sum_{i,j}\frac{1}{2}\phi_{i,j}^{2}(t=0), \tag{6.4}$$

where M is the number of grid-points.

The initial variance is determined by the initial distribution of the scalar. We have chosen an axi-symmetric profile with zero mean and linear in the distance to the domain centre.

The Crank–Nicholson scheme in Eq. (6.3), the leap-frog scheme in Eq. (5.2), the two-step Adams–Bashforth scheme in Eq. (5.3) and the ADI scheme in Eqs. (5.4) and (5.5) are tested and compared with a mixed second-order central – third-order upwind ADI scheme, that is currently used in our shallow water solver for scalar transport. This scheme is defined below, see also [6].

First-half time step, implicit third-order upwind in x-direction, explicit second-order central in y-direction:

$$\frac{\phi^{n+1/2}-\phi^{n}}{\frac{1}{2}\Delta t}+UD_x^{\text{3rd upw}}\left\{\phi^{n+1/2}\right\}+VD_y^{\text{2nd central}}\{\phi^{n}\}=0, \tag{6.5}$$

second-half time step, implicit third-order upwind in y-direction, explicit second-order central in x-direction:

$$\frac{\phi^{n+1}-\phi^{n+1/2}}{\frac{1}{2}\Delta t}+UD_x^{\text{2nd central}}\left\{\phi^{n+1/2}\right\}+VD_y^{\text{3rd upw}}\{\phi^{n+1}\}=0, \tag{6.6}$$

where D_x and D_y are difference operators with respect to x and y, respectively.

To maintain a clear distinction between this ADI scheme, that uses either a second-order central or a third-order upwind scheme for the convection terms and the one we defined in the previous section in Eqs. (5.4) and (5.5), in the following we refer to these two schemes as 'third-order upwind ADI' and 'SS ADI' (skew-symmetric ADI), respectively.

Fig. 3 presents the temporal development of the scalar-field during the test without physical diffusion. At two instants (the initial condition and at $time = 2$, see Eq. (6.9)) in time this figure shows snapshots taken during the simulation.

Fig. 4 shows the development of the variance $\mathrm{var}(t)/\mathrm{var}(0)$ in the computational domain for the five different space–time iterations:

- We see that CN (the Crank–Nicholson integration method – space–time variance-conserving scheme) produces a straight line at a value of $\mathrm{var}(t)/\mathrm{var}(0) = 1$.
- The explicit LF scheme (leap-frog) wiggles around a value slightly higher than unity, which can be explained by the following.

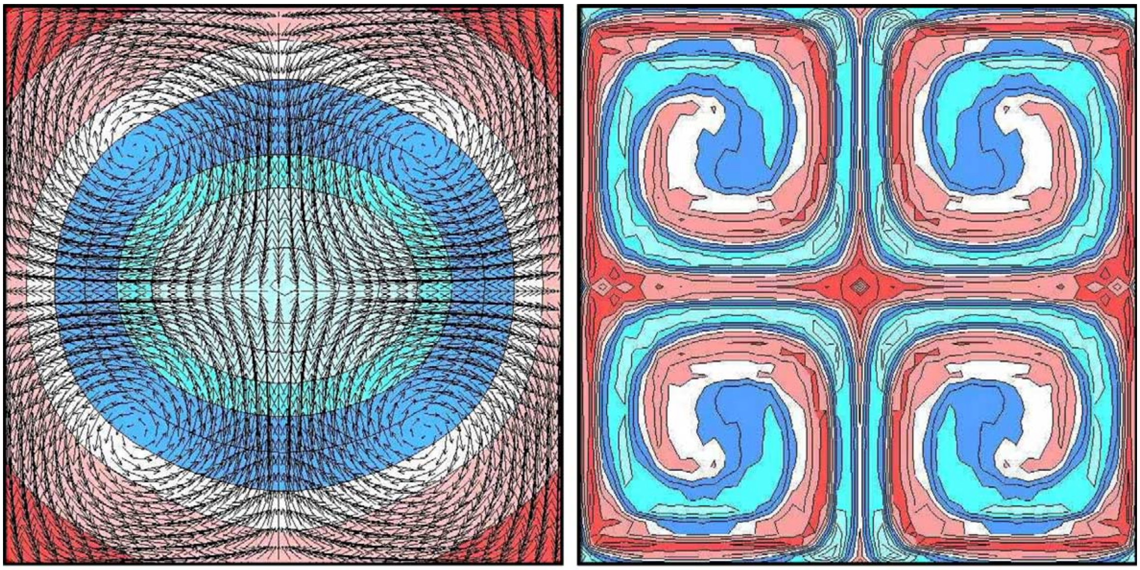

Fig. 3. Snapshots of the scalar distribution and the frozen velocity field during the simulation at dimensionless time (see Eq. (6.9)): (a) time = 0 and (b) time = 2.

Consider the semi-discrete formulation of leap-frog temporal integration:

$$\frac{\phi^{(n+1)} - \phi^{(n-1)}}{2\Delta t} = \vec{u} \cdot \vec{\nabla}\phi^{(n)}. \tag{6.7}$$

Assuming the RHS of (6.7) is discretized by a spatially variance-conserving advection scheme then pre-multiplication of (6.7) by $\phi^{(n)}$ and subsequently integration over a closed volume yields

$$\int \phi^{(n)}\phi^{(n+1)} - \phi^{(n)}\phi^{(n-1)}\mathrm{d}V = 0. \tag{6.8}$$

A class of solutions obeying (6.8) reads $\phi^{(n)} = a(\underline{x}) + (-1)^n b(\underline{x})$ so that $\{\phi^{(n)}\}^2 = a^2(\underline{x}) + b^2(\underline{x}) + 2(-1)^n a(\underline{x})b(\underline{x})$ holds. The latter and possibly also its volume-integral yields an oscillatory energy level that may be the cause of the wiggling energy levels in Fig. 4 for the leap-frog scheme.

- The SS ADI scheme also deviates a little from unity.
- The mixed central-third-order upwind ADI scheme: all three time integration methods mentioned above conserve variance far better than the mixed central-third-order upwind ADI scheme.
- The AB (two-step Adams–Bashforth scheme) becomes unstable after approximately one dimensionless time unit, for a definition see Eq. (6.9), which illustrates that the use of a spatially variance-conserving scheme does not guarantee a stable simulation; one definitely has to take into account the properties of the time integration method.

Fig. 4 presents $\mathrm{var}(t)/\mathrm{var}(0)$ as a function of the dimensionless time, defined by

$$\mathrm{time} = \frac{n\Delta t}{L/U_0}, \tag{6.9}$$

where n is the number of time steps, L is the width of the computational domain and U_0 the amplitude of the velocity, see Eq. (6.2). We found that one dimensionless time unit (6.9) is proportional to one full rotation

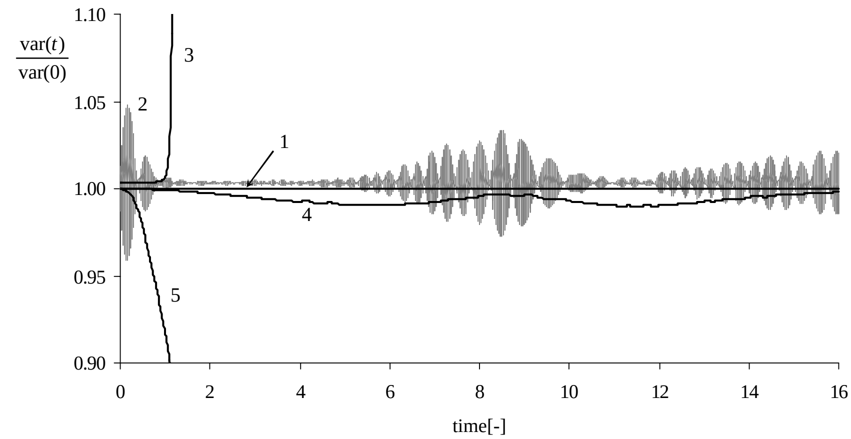

Fig. 4. Development of variance for cfl = 0.8 for: (1) CN, (2) LF, (3) AB, (4) SS ADI, (5) 3rd upw ADI.

of a fluid particle in a vortex. For the Courant number, $cfl = U_0 \cdot \Delta t/\Delta x = 0.8$ holds in the computations presented in Fig. 4.

In Table 1 some properties are given to illustrate the differences between four of the five space–time-iterations in the simulation of Fig. 4. In this table, the following distinction is made: first, the two different spatial discretizations (SS ADI and third-order upwind ADI) are compared. Second, a comparison between the three different time iteration methods (CN, LF and SS ADI) is made. The two-step Adams–Bashforth method is omitted in this table, because it is unstable for pure advection. The properties that are presented in Table 1 are

$$\overline{\Delta V} = \frac{\sum_{n=1}^{n_{\max}} [\mathrm{var}(n\Delta t) - \mathrm{var}(0)]/\mathrm{var}(0)}{n_{\max}}, \tag{6.10}$$

$$\Delta V_{\max} = \max\{|\mathrm{var}(t) - \mathrm{var}(0)|/\mathrm{var}(0)\}, \tag{6.11}$$

where $n_{\max}$ is the number of time steps in the simulation.

Fig. 4 and Table 1 show that the strictly variance-conserving Crank–Nicholson scheme has indeed far better conservation properties than the other schemes.

To illustrate that the conservation of variance of CN is independent of the time-step used, Fig. 5 presents $|\overline{\Delta V}|$ and $\Delta V_{\max}$ for CN for three different time steps on a double logarithmic scale. Also, a line proportional to Δt^2 is drawn, because the Crank–Nicholson time-stepping method used in the variance-conserving scheme is second-order in time. Obviously, there is no correlation between the lines representing $|\overline{\Delta V}|$ and Δt^2, nor between $\Delta V_{\max}$ and Δt^2.

Table 1
Values given in Eqs. (6.10) and (6.11) for four time-iteration methods for $cfl = 0.8$

	$\overline{\Delta V}$	$\Delta V_{\max}$	
SS ADI	−0.00588	0.01008	Same time-integration, different spatial discretization
3rd upw	−0.83140	0.9870	
CN	1.006×10^{-14}	2.580×10^{-14}	Same spatial discretization, different time-integration
LF	0.00324	0.04770	
SS ADI	−0.00588	0.01008	

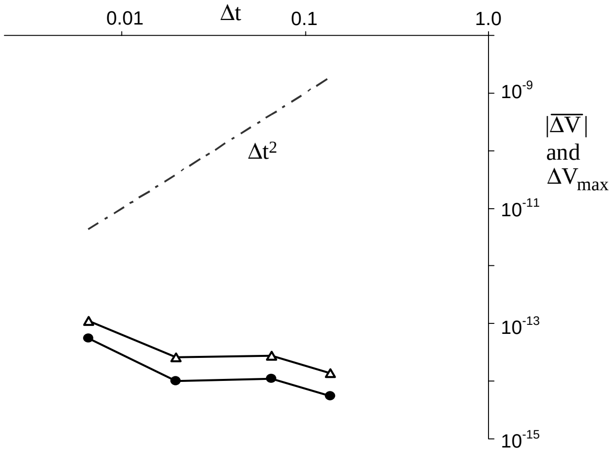

Fig. 5. Variance conservation as a function of time-step for CN: ● = $|\overline{\Delta V}|$; $\Delta = \Delta V_{max}$.

From this section we can conclude that our space–time variance-conserving scheme (4.14) is indeed strictly variance-conserving, i.e. there is no correlation between the time-step and the violation of variance conservation.

In the following section, we discuss another interesting and important property of the skew-symmetric and the space–time variance-conserving convection schemes.

7. Conditions for the consistency of the skew-symmetric and the space–time variance-conserving convection schemes

In the following, we will discuss the consistency of the skew-symmetric and the space–time variance-conserving convection schemes. We will show that these schemes are not unconditionally consistent.

The skew-symmetric scheme as defined in Eq. (3.1) and the variance-conserving scheme as defined in Eq. (4.12) are not unconditionally consistent. This can be shown by Taylor expansions of both expressions.

First consider the Taylor expansion of the expression in Eq. (3.1), with respect to (i,j):

$$
\begin{aligned}
(Skew)_{\Delta,(i,j)} &\equiv \frac{u_{i+1/2,j}\phi_{i+1,j} - u_{i-1/2,j}\phi_{i-1,j}}{2\Delta x} + \frac{v_{i,j+1/2}\phi_{i,j+1} - v_{i,j-1/2}\phi_{i,j-1}}{2\Delta y} \\
&= u\frac{\partial\phi}{\partial x} + v\frac{\partial\phi}{\partial y} + \frac{1}{2}\phi\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] + \mathrm{O}(\Delta x^2, \Delta y^2). \qquad (7.1)
\end{aligned}
$$

Eq. (7.1) shows that the skew-symmetric scheme is consistent only if $\vec{\nabla}\cdot\vec{u} = 0$ holds.

For the variance-conserving scheme in one dimension we have the following:

$$
(Conv)_{\Delta,i}\frac{u_{i+1/2}\left(\phi_{i+1}^{n+1} + \phi_{i+1}^{n}\right) - u_{i-1/2}\left(\phi_{i-1}^{n+1} + \phi_{i-1}^{n}\right)}{4\Delta x} = u\frac{\partial\phi}{\partial x} + \frac{1}{2}\phi\frac{\partial u}{\partial x} + \mathrm{O}(\Delta x^2, \Delta t^2) \qquad (7.2)
$$

Eq. (7.2) shows that for one dimension, the variance-conserving scheme cannot be consistent, because in general $\frac{1}{2}\phi\frac{\partial u}{\partial x} \neq 0$ holds. For consistency, we need a multi-dimensional formulation that can guarantee zero divergence, i.e. $\vec{\nabla}\cdot\vec{u} = 0$. The same goes for the skew-symmetric scheme in Eq. (3.1).

From the previous, we can conclude that the skew-symmetric scheme and the time–space variance-conserving scheme are consistent only for flows where $\vec{\nabla}\cdot\vec{u} = 0$ holds, i.e. flows that are multi-dimensional and that are divergence-free in the continuous formulation. The absence of divergence on the discrete level is not necessary for consistency, nor for the conservation of variance, as stated before in the argumentation below Eq. (2.9).

This concludes the derivation of the scheme based on variance conservation of a scalar property.

In the next section, we present the space–time kinetic energy-conserving scheme for the inviscid Burgers' equation. In our opinion, we present a clearer definition of the kinetic energy conserving scheme than Morinishi et al. [5], thanks to the clear distinction between the flux terms and the convected quantity.

8. The space–time kinetic energy-conserving scheme

In this section, we will extend the strictly variance-conserving scheme as defined in Eq. (4.14) to an equivalent scheme for the convection terms in the Navier–Stokes equations.

The skew-symmetric form for the convection terms in the Navier–Stokes equations results from Fig. 1, by shifting the computational cell $\frac{1}{2}\Delta x$ for the x-momentum equation and $\frac{1}{2}\Delta y$ for the y-momentum equation, and applying spatial central averaging:

x-momentum:

$$\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)_{i+1/2,j}\approx (Skew(x))_{\Delta,(i+1/2,j)} = \frac{\bar{u}^x_{i+1,j}u_{i+3/2,j}-\bar{u}^x_{i,j}u_{i-1/2,j}}{2\Delta x}+\frac{\bar{v}^x_{i+1/2,j+1/2}u_{i+1/2,j+1}-\bar{v}^x_{i+1/2,j-1/2}u_{i+1/2,j-1}}{2\Delta y} \tag{8.1}$$

y-momentum:

$$\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)_{i,j+1/2}\approx (Skew(y))_{\Delta,(i,j+1/2)} = \frac{\bar{u}^y_{i+1/2,j+1/2}v_{i+1,j+1/2}-\bar{u}^y_{i-1/2,j+1/2}v_{i-1,j+1/2}}{2\Delta x}+\frac{\bar{v}^y_{i,j+1}v_{i,j+3/2}-\bar{v}^y_{i,j}v_{i,j-1/2}}{2\Delta y}. \tag{8.2}$$

The average values – interpreted as fluxes through the boundaries of the computational cell, see Fig. 6 – are defined as follows:

$$\bar{\alpha}^x_{i,j}=\frac{1}{2}(\alpha_{i+1/2,j}+\alpha_{i-1/2,j}),\quad \bar{\alpha}^y_{i,j}=\frac{1}{2}(\alpha_{i,j+1/2}+\alpha_{i,j-1/2}). \tag{8.3}$$

Fig. 6(a) and (b) present the computational cells for the convection terms in the x and y-momentum equations.

As derived before in Section 4, the time level of the flux terms does not have to be specified for creating a kinetic energy-conserving convection scheme. Therefore, the kinetic energy-conserving scheme for what is now the two-dimensional inviscid Burgers' equation is defined as

$$\frac{u^{n+1}_{i+1/2,j}-u^n_{i+1/2,j}}{\Delta t}+\frac{\left(\bar{u}^x_{i+1,j}\right)^\theta\left(u^{n+1}_{i+3/2,j}+u^n_{i+3/2,j}\right)-\left(\bar{u}^x_{i,j}\right)^\theta\left(u^{n+1}_{i-1/2,j}+u^n_{i-1/2,j}\right)}{4\Delta x} + \frac{\left(\bar{v}^x_{i+1/2,j+1/2}\right)^\theta\left(u^{n+1}_{i+1/2,j+1}+u^n_{i+1/2,j+1}\right)-\left(\bar{v}^x_{i+1/2,j-1/2}\right)^\theta\left(u^{n+1}_{i+1/2,j-1}+u^n_{i+1/2,j-1}\right)}{4\Delta y}=0, \tag{8.4}$$

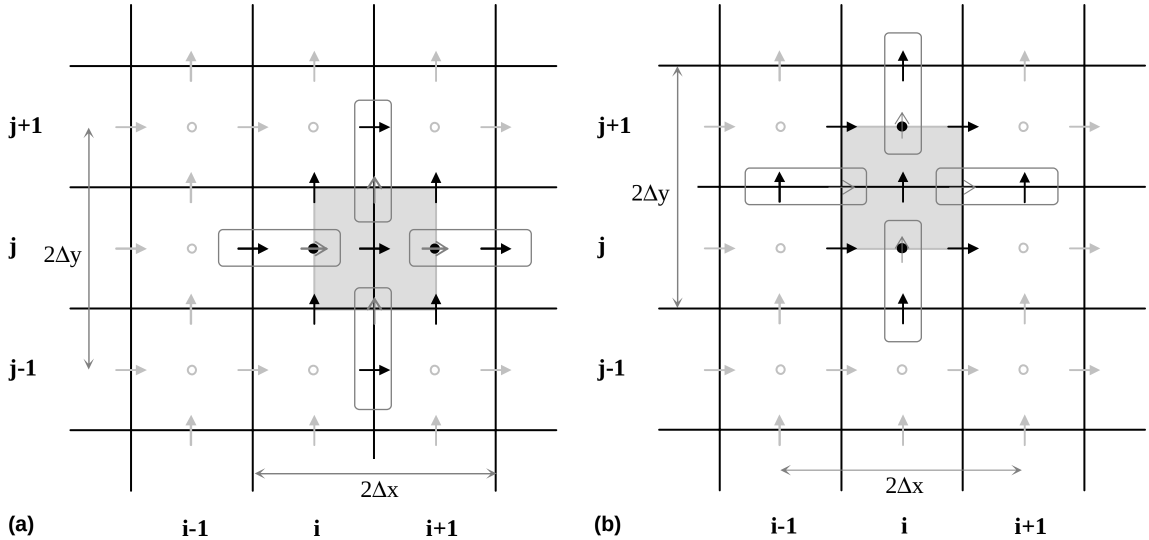

Fig. 6. (a) Computational cell of skew-symmetric convection term for x-momentum equation. (b) Computational cell of skew-symmetric convection term for y-momentum equation.

$$\frac{v^{n+1}_{i,j+1/2}-v^{n}_{i,j+1/2}}{\Delta t}+\frac{\left(\bar{u}^{y}_{i+1/2,j+1/2}\right)^{\theta}\left(v^{n+1}_{i+1,j+1/2}+v^{n}_{i+1,j+1/2}\right)-\left(\bar{u}^{y}_{i-1/2,j+1/2}\right)^{\theta}\left(v^{n+1}_{i-1,j+1/2}+v^{n}_{i-1,j+1/2}\right)}{4\Delta x}$$
$$+\frac{\left(\bar{v}^{y}_{i,j+1}\right)^{\theta}\left(v^{n+1}_{i,j+3/2}+v^{n}_{i,j+3/2}\right)-\left(\bar{v}^{y}_{i,j}\right)^{\theta}\left(v^{n+1}_{i,j-1/2}+v^{n}_{i,j-1/2}\right)}{4\Delta y}=0, \tag{8.5}$$

where the time level θ can be chosen arbitrarily, but consistently.

9. Conclusions

We can recapitulate our findings as follows:

A strictly (space–time) variance-conserving, or kinetic-energy conserving scheme for the convection terms contains the same time levels as the ones used in the time-derivative term. This means that the following must hold:

$$\frac{\phi^{p}_{i,j}-\phi^{q}_{i,j}}{(p-q)\Delta t}=-\left(\frac{1}{2}C^{p}_{i,j}+\frac{1}{2}C^{q}_{i,j}\right), \tag{9.1}$$

where $C^{k}_{i,j}$ is the skew-symmetric scheme for the convection term for a scalar quantity, as proposed by Piacsek and Williams [1], but with time levels assigned to the convected quantity:

$$C^{k}_{i,j}=\frac{u^{\theta}_{i+1/2,j}\phi^{k}_{i+1,j}-u^{\theta}_{i-1/2,j}\phi^{k}_{i-1,j}}{2\Delta x}+\frac{v^{\theta}_{i,j+1/2}\phi^{k}_{i,j+1}-v^{\theta}_{i,j-1.2}\phi^{k}_{i,j-1}}{2\Delta y}, \tag{9.2}$$

or the skew-symmetric scheme for the convection term for a vector quantity, see e.g. [5], with time-levels assigned to the convected quantity

$$C^k_{x(i+1/2,j)} = \frac{\left(\bar{u}^x_{i+1,j}\right)^\theta u^k_{i+3/2,j} - \left(\bar{u}^x_{i,j}\right)^\theta u^k_{i-1/2,j}}{4\Delta x} + \frac{\left(\bar{v}^x_{i+1/2,j+1/2}\right)^\theta u^k_{i+1/2,j+1} - \left(\bar{v}^x_{i+1/2,j-1/2}\right)^\theta u^k_{i+1/2,j-1}}{4\Delta y},$$
$$C^k_{y(i,j+1/2)} = \frac{\left(\bar{u}^y_{i+1/2,j+1/2}\right)^\theta v^k_{i+1,j+1/2} - \left(\bar{u}^y_{i-1/2,j+1/2}\right)^\theta v^k_{i-1,j+1/2}}{4\Delta x} + \frac{\left(\bar{v}^y_{i,j+1}\right)^\theta v^k_{i,j+3/2} - \left(\bar{v}^y_{i,j}\right)^\theta v^k_{i,j-1/2}}{4\Delta y}. \tag{9.3}$$

Here the time-level θ is not prescribed and can be chosen arbitrarily but consistently, which has the advantage that the discrete equations of the non-linear convection terms yield a linear system to solve.

Considering the definition of the material derivative

$$\frac{\mathrm{D}}{\mathrm{D}t}\phi = \left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right)\phi \approx \frac{\phi^p_{i,j} - \phi^q_{i,j}}{(p-q)\Delta t} + \left(Conv^{(p+q)/2}_{\Delta,(i,j)}\right)\phi = 0, \tag{9.4}$$

where $(Conv^{(p+q)/2}_{\Delta,(i,j)})\phi$ is the approximation of $(\vec{u}\cdot\vec{\nabla})$ at time-level $(p+q)/2$ and at $(x,y) = (i,j)$ in space, it is no surprise that the time levels of the convected quantity ϕ have to be the same in the time term $\partial/\partial t$ and the convective term $\vec{u}\cdot\vec{\nabla}$ and that the time levels of the flux term $\vec{u}$ can be chosen.

Our space–time variance-conserving scheme (4.12) is strictly variance-conserving, i.e. there is no correlation between the time-step and the violation of variance conservation. The leap-frog scheme wiggles around a value slightly higher than unity for the non-diffusive problem tested in Section 6. However, in practical problems one will always have to deal with diffusion or dissipation and in this case the leap-frog scheme becomes unusable. The two-step Adams–Bashforth scheme becomes unstable for the non-diffusive problem. For diffusive problems, the two-step Adams–Bashforth scheme can be stable, but it has a time-step restriction. The SS ADI scheme is a compromise between the strictly variance-conserving Crank–Nicholson scheme and the mixed central-third-order upwind ADI scheme. It has fairly good conservation properties and it is unconditionally stable.

According to Section 7, the variance-conserving scheme is second order in space for equidistant grids. In the manner of Richardson extrapolation, see also [4] and [5], higher order schemes can be constructed. Furthermore, our analysis applies to equidistant grids only. When the variance-conserving scheme is used on non-equidistant grids, we either loose space-accuracy while maintaining strict variance conservation, or introduce errors in variance conservation.

10. A final consideration

In this paper, we showed that the strictly variance-conserving Crank–Nicholson scheme is the best choice when it variance conservation is important. In the civil engineering problems however, the following considerations are essential.

In the shallow-water simulations, two Courant numbers play a dominant role. For a consistent space–time representation of interacting vortices, the Courant number C_U for advection should be less than unity. The latter upper limit allows for an explicit temporal integration. In nearly all applications, however, the Froude number (F) is significantly less than unity so that the Courant number C_{BT} for barotropic or long-wave modes easily exceeds unity ($C_{BT} = C_U \cdot F^{-1}$). Consequently, the large C_{BT} (>1) requires an implicit coupling between the conservation equation, expressed in the free-surface elevation, and the related hydrostatic-pressure gradient in the horizontal momentum equations, for details of this procedure see [7] and references therein. Solving these coupled conservation and momentum equations with a Crank–Nicholson scheme introduces a highly complex stencil, whereas an ADI temporal integration method yields a much more simple tridiagonal system. With this in mind, and also because of our experience with ADI, which is the method currently used in our shallow-water solver, we consider ADI as a computationally optimal compromise of the ultimate variance-conserving method that we present in this paper.

Acknowledgements

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